

Temperature of a single chaotic eigenstate

Fausto Borgonovi,^{1,2} Francesco Mattiotti,¹ and Felix M. Izrailev^{3,4}

¹*Dipartimento di Matematica e Fisica and Interdisciplinary Laboratories for Advanced Materials Physics, Università Cattolica, via Musei 41, 25121 Brescia, Italy*

²*Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via Bassi 6, I-27100 Pavia, Italy*

³*Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla 72570, Mexico*

⁴*NSCL and Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824-1321, USA*

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The onset of thermalization in a closed system of randomly interacting bosons at the level of a single eigenstate is discussed. We focus on the emergence of Bose-Einstein distribution of single-particle occupation numbers, and we give a local criterion for thermalization dependent on the eigenstate energy. We show how to define the temperature of an eigenstate, provided that it has a chaotic structure in the basis defined by the single-particle states. The analytical expression for the eigenstate temperature as a function of both interparticle interaction and energy is complemented by numerical data.

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I. INTRODUCTION

The subject of thermalization occurring in isolated quantum systems of interacting particles has been developed in the last decades due to various applications in nuclear and atomic physics [1,2], as well as in view of basic problems of statistical mechanics [3–5]. Recently, the interest in this subject has increased due to experiments with cold atoms and molecules in optical lattices [6] and trapped ions [7]. Correspondingly, many theoretical and numerical studies have been performed in order to understand the mechanism of thermalization in the absence of a heat bath (see [5] and references therein). Nevertheless, despite some studies about the onset of thermalization as a function of various physical parameters such as the number of particles [8], the strength of interparticle interaction [9], and the choice of initially excited states [10], the role of these items still remains open.

The mechanism driving thermalization in isolated systems of interacting particles is associated with quantum chaos [11]. Different from the well developed one-body chaos theory, many problems related to many-body chaos, such as the thermalization of Fermi and Bose particles, are currently under intensive study. Unlike classical chaos, which is intrinsically related to the instability of motion with respect to a change in initial conditions, quantum chaos manifests itself in specific fluctuations of the energy spectra and in the chaotic structure of eigenstates. As shown in [9], the properties of the energy spectra are less important to the statistical relaxation toward a steady-state distribution than the structure of the eigenstates in the physically chosen many-particle basis. Therefore, the main interest in the study of many-body chaos was shifted long ago to the properties of many-body eigenstates.

Chaotic eigenstates play a key role in the statistical description of isolated quantum systems. As stressed long ago [12], conventional statistical mechanics can be established on the level of individual quantum states and not only by averaging over many states. This was confirmed numerically decades ago (see, for instance, [11] and discussion in [13]). However, this fact has no practical consequences unless the conditions for such a situation are developed. One of the open problems in this field is to

establish these conditions for systems with a finite number of particles.

To date, many problems have been addressed concerning the problem of thermalization in isolated systems. Here we raise a new one which is directly related to this issue. It is already agreed that one can speak of thermalization on the level of an individual state, and various characteristics of thermalization have been under extensive study, such as the relaxation of a system to steady state distributions after various quenches, decay of correlations in time for observables and their fluctuations after relaxation, etc. [6–10,14].

Now, in view of the basic concepts of statistical mechanics and recent experiments with cold atoms and molecules [6] it is natural to ask a question about the onset of the Bose-Einstein distribution (BED) emerging on the level of a *single* many-body eigenstate due to the interaction between bosons and not to an external field or a thermostat. Below we specifically initiate the study of the onset of BED in a finite system of interacting bosons, which is expected to occur when the interparticle interaction is strong enough. We suggest a semianalytical approach able to reveal the conditions under which an isolated many-body eigenstate can be considered thermal and introduce its temperature in relation to the model parameters.

II. THE MODEL AND BASIC CONCEPTS

At variance with eigenvalues, many-particle eigenstates are defined by means of a suitable single-particle basis. The latter, for its part, has direct relevance to physical reality, specifically, to the choice of the mean field to which quantum observables such as occupation numbers are referred. Correspondingly, we assume that the total Hamiltonian H can be presented as the sum of the mean field H_0 describing noninteracting (quasi)particles and a residual interaction V , modeled as a two-body random interaction. Such a setup, based on a random interaction, also serves as a good model for a deterministic interaction between bosons [14], where the complexity in many-body matrix elements emerges due to the complicated nature of the interaction itself.

In this paper we consider N identical bosons occupying M single-particle levels specified by random energies ϵ_s with mean spacing, $\langle \epsilon_s - \epsilon_{s-1} \rangle = 1$. Let us notice that the randomness in the single-particle spectrum is not strictly necessary for the results obtained: it has been introduced only in order to avoid the degeneracies in the unperturbed many-body spectrum.

The Hamiltonian reads

$$H = H_0 + V = \sum_s \epsilon_s a_s^\dagger a_s + \sum_{s_1 s_2 s_3 s_4} V_{s_1 s_2 s_3 s_4} a_{s_1}^\dagger a_{s_2}^\dagger a_{s_3} a_{s_4}, \quad (1)$$

where the two-body matrix elements $V_{s_1 s_2 s_3 s_4}$ are random Gaussian entries with zero mean and variance V^2 . The dimension of the Hilbert space generated by the many-particle basis states is $N_H = (N + M - 1)!/N!(M - 1)!$. Here we consider $N = 6$ particles in $M = 11$ levels (dilute limit, $N \leq M$) for which $N_H = 8008$.

Two-body random interaction (TBRI) matrices (1) have a quite long history. They were introduced in [15] and extensively studied for fermions [16]. On the other hand, the case of Bose particles has been less investigated, and only a few results are known, typically for the dense limit, $N \gg M$ [17,18].

The eigenstates of H can be generically represented in terms of the basis states $|k\rangle = a_{k_1}^\dagger \cdots a_{k_N}^\dagger |0\rangle$, which are eigenstates of H_0 ,

$$|\alpha\rangle = \sum_k C_k^\alpha |k\rangle, \quad (2)$$

where it has been implicitly assumed that

$$H|\alpha\rangle = E^\alpha |\alpha\rangle \quad (3)$$

and

$$H_0|k\rangle = E_k^0 |k\rangle. \quad (4)$$

A characterization of the number of principal components C_k^α in an eigenstate $|\alpha\rangle$ can be obtained by the study of the participation ratio,

$$N_{pc} = 1 / \sum_k |C_k^\alpha|^4. \quad (5)$$

If the number N_{pc} of the principal components C_k^α is sufficiently large (we will specify later how large it should be) and C_k^α can be considered to be random and noncorrelated, this is the case of *chaotic* eigenstates. This notion is quite different from full delocalization in the unperturbed basis since for isolated systems the eigenstates typically fill only a part of the unperturbed basis [11].

To characterize the structure of the eigenstates, we use the *F function*,

$$F^\alpha(E) = \sum_k |C_k^\alpha|^2 \delta(E - E_k^0), \quad (6)$$

which is the energy representation of an eigenstate. From the components C_k^α one can also construct the *strength function* (SF) of a basis state $|k\rangle$,

$$F_k(E) = \sum_\alpha |C_k^\alpha|^2 \delta(E - E^\alpha), \quad (7)$$

widely used in nuclear physics [19] and known in solid state physics as the *local density of states*. The SF shows how the basis state $|k\rangle$ decomposes into the exact eigenstates $|\alpha\rangle$ due to the interaction V . It can be measured experimentally, and it is of great importance since its Fourier transform gives the time evolution of an excitation initially concentrated in the basis state $|k\rangle$. Specifically, it defines the survival probability to find the system at time t in the initial state $|k\rangle$.

On increasing the interaction strength, the SF in isolated systems undergoes a crossover from a δ -like function (perturbative regime) to a Breit-Wigner (BW), with a width well described by Fermi's golden rule. With a further increase in the interaction, the form of the SF tends to a Gaussian [3,9,20], a scenario that has also been observed experimentally [21].

One of the basic concepts in our approach is the so-called *energy shell*, which is the energy region defined by the projection of V onto the basis of H_0 [22]. This region is the largest one that can be occupied by an eigenstate. The partial filling of the energy shell by an eigenstate can be associated with the many-body localization in the energy representation, a subject that is nowadays under intensive investigation (see, for example, [23] and references therein). When this happens, of course, the eigenstates cannot be treated as thermal, in the sense that a good definition of temperature cannot be done. Contrary to this, if a chaotic eigenstate fills the energy shell completely, this corresponds to maximal quantum chaos, and a proper temperature can be defined.

In the past a parameter driving the global crossover from nonchaotic to chaotic eigenstates was proposed based essentially on the ratio between the interaction strength and the mean energy range spanned by the basis states effectively coupled by the interaction V [3,4,11]. This criterion is independent of the energy of the eigenstate. Since we are dealing here with single eigenstates, we will generalize this idea in order to obtain a local criterion (i.e., depending on the eigenenergy) for such a crossover.

Each many-body eigenstate $|\alpha\rangle$ is characterized not only by an "effective number" of occupied basis states N_{pc} , i.e., a number of principal components in the unperturbed basis, but also by an unperturbed energy width,

$$\delta_0 = (\langle \alpha | H_0^2 | \alpha \rangle - \langle \alpha | H_0 | \alpha \rangle^2)^{1/2}. \quad (8)$$

These two parameters allow us to define, for each single eigenstate, an effective mean energy spacing, $d_{loc} = \delta_0 / N_{pc}$, that the perturbation strength V must overcome in order to go beyond the perturbative regime. Accordingly, in order to have chaotic eigenstates we require $V > d_{loc}$, while for $V < d_{loc}$ we can speak of a perturbative regime. In the following we will see that the region characterized by $V > d_{loc}$ is the "thermal" one, where an effective temperature, dependent on the interparticle interaction, can be defined via the Bose-Einstein distribution.

III. THE BOSE-EINSTEIN DISTRIBUTION FOR AN INTERACTING EIGENSTATE

In order to define the temperature for each selected eigenstate $|\alpha\rangle$ let us consider its occupation number distribution

(OND),

$$n_s^\alpha = \langle \alpha | \hat{n}_s | \alpha \rangle = \sum_k |C_k^\alpha|^2 \langle k | \hat{n}_s | k \rangle. \quad (9)$$

As one can see, the OND (9) consists of two ingredients: the probabilities $|C_k^\alpha|^2$ and the occupation numbers $\langle k | \hat{n}_s | k \rangle$ related to the basis states of H_0 . The latter are just integer numbers $0, 1, 2, \dots, N$ depending on how many bosons occupy the single-particle level s with respect to the many-body state $|k\rangle$. If the eigenstate $|\alpha\rangle$ of H consists of many uncorrelated components, one can substitute $|C_k^\alpha|^2$ by the corresponding SF obtained by an average either over a number of eigenstates with close energies or *inside an individual eigenstate*, for example, with the use of the “moving window” average [11]. Thus, from the knowledge of the SF it is possible to obtain the OND without the diagonalization of huge Hamiltonian matrices.

Having defining the temperature of a single eigenstate by means of its corresponding OND in mind, a few relevant questions come out. First of all, since we are dealing with bosons, the common reference OND is the BED that is derived for noninteracting particles in the thermodynamic limit. The situation here is clearly different since our system has a finite number of interacting particles. To address this question properly we start with the basic relations

$$\sum_s n_s = N, \quad \sum_s \epsilon_s n_s = E, \quad (10)$$

where N is the total number of bosons and E is the energy of a system for which the interparticle interaction is neglected. As is known, the solution of these equations for $N \rightarrow \infty$ leads to the BED,

$$n_s^{BE} = (e^{\beta(\epsilon_s - \mu)} - 1)^{-1}. \quad (11)$$

The derivation can be obtained due to only the combinatorics, with the constants β and μ being the Lagrange multipliers [24] (see also discussion in Ref. [25]). The meaning of β and μ as the inverse temperature and chemical potential, respectively, emerges when the system is connected with a heat bath. However, we will show that one can speak of BED even if the system is isolated; moreover, this distribution emerges on the level of a single eigenstate of the total Hamiltonian. Inserting (11) into (10), one can obtain both β and $z = e^{\beta\mu}$ as a function of N and E . If we further fix the number of particles N , we obtain two functions, $z(E)$ and $\beta(E)$, as shown in Fig. 1. The values of z and β corresponding to the energy E^α are indicated in Fig. 1 by open circles that are obtained with the intersection of the vertical line $E = E^\alpha$ with the curves $z(E)$ and $\beta(E)$. Let us note that the BED indicated by a dashed line in Fig. 2(d) has been obtained using exactly these values of z and β .

Now the key question is, what energy on the right-hand side of Eq. (10) should be used for interacting bosons in order to have, if any, the correspondence to the numerically obtained OND [26]?

First, we start with the global correspondence between the actual OND numerically obtained from individual eigenstates (9) and the BED expression (11). For this we consider the OND averaged over a number of close eigenstates in a narrow energy window. We considered the average over a small energy window with the only purpose being to study fluctuations in

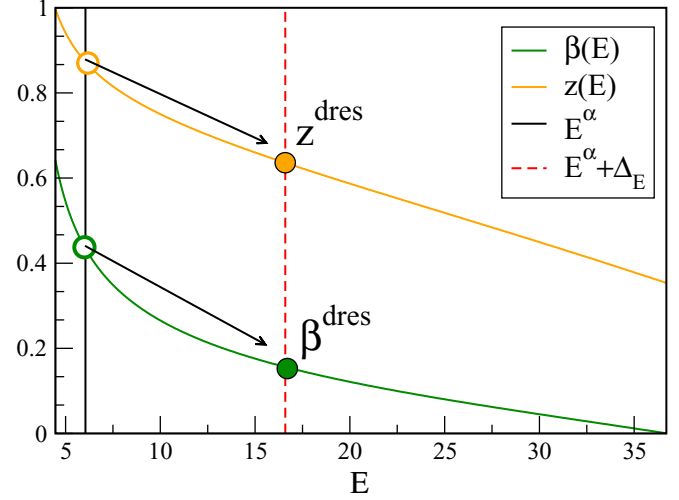


FIG. 1. Pictorial description of the increase of temperature for the eigenstate in Fig. 2(d).

the next section. In any case the ONDs for single eigenstates are shown in Fig. 6 in Appendix A.

We choose the eigenenergy E^α in two different regions, close to the center and close to the edges of the spectrum, and, for each of them, two different values of the interaction strength V (see Fig. 2). Each panel of Fig. 2 shows the average values of the ONDs, with the error bars representing one standard deviation (fluctuations here are due to different eigenstates in a close energy window; alternatively, one can choose one single eigenstate and change the random interparticle potential), and two curves. The dashed curves are those obtained by choosing as E_α the unperturbed energy, while the solid ones are obtained by “dressing” the energy, as shown below.

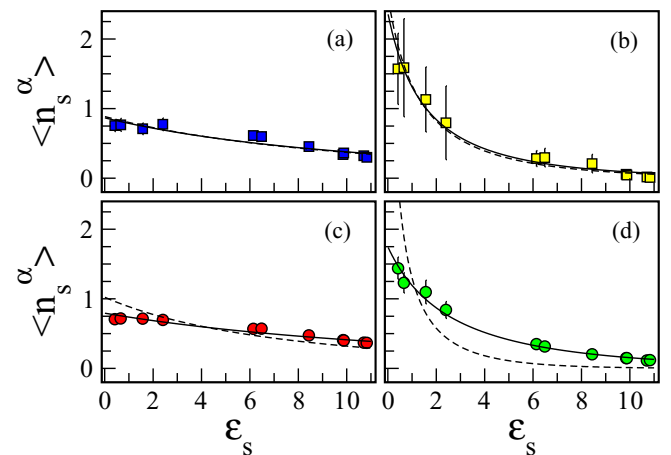


FIG. 2. Average occupation numbers $\langle n_s^\alpha \rangle$ for weak ($V = 0.1$, top panels) and strong ($V = 0.4$, bottom panels) perturbation and different energies in the middle of the energy spectrum, (a) $E^\alpha = 28.51$ and (c) $E^\alpha = 25.93$, and on the edges, (b) $E^\alpha = 11.27$ and (d) $E^\alpha = 6.05$. Error bars indicate one standard deviation and are obtained by averaging over 20 close eigenstates. Dashed curves are obtained from the BED with $E = E^\alpha$ in Eq. (10). Solid curves correspond to the BED with the energy E^{dres} in Eq. (12).

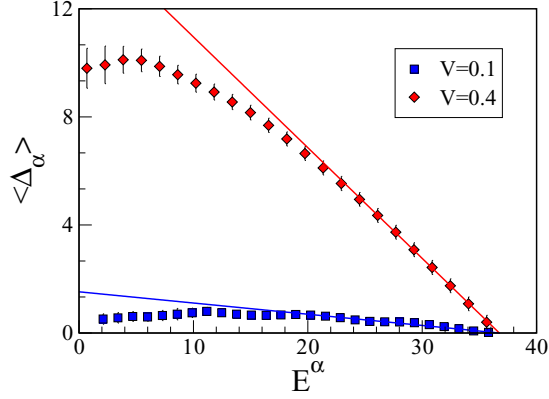


FIG. 3. Average energy shift $\langle \Delta_\alpha \rangle$ as a function of the energy E^α for two different values of the interaction V . Symbols stand for numerical results, while solid lines represent the Gaussian approximation (error bars indicate one standard deviation). The average has been done over 20 close eigenstates. Due to the symmetry only half of the energy spectrum (where the density of states is an increasing function of the energy) is shown.

As one can see in Fig. 2, while for weak interaction (top panels) the dashed lines match perfectly the numerical data, this does not happen for strong interaction (bottom panels). While such a failure in the case of strong interaction is not unexpected, the good agreement in the case of weak interaction is far from trivial since it is worth noting that the Bose-Einstein distribution is obtained in the limit $N \rightarrow \infty$ while here we have only $N = 6$ particles.

To take into account the interparticle interaction we use the approach suggested in Refs. [3,4]. Specifically, we substitute the energy $E = E^\alpha$ in (10) with the dressed energy,

$$E^{\text{dres}} = \langle \alpha | H_0 | \alpha \rangle \equiv E^\alpha + \Delta_\alpha. \quad (12)$$

Note that this energy is higher [in the region in which the density of states (DOS) increases with energy] than the eigenvalue E^α corresponding to the eigenstate $|\alpha\rangle$. This corresponds to a temperature T^{dres} higher than that obtained with the substitution $E \rightarrow E^\alpha$. The dressed energy $E^{\text{dres}} = E^\alpha + \Delta_\alpha$ is indicated by a vertical dashed line in Fig. 1. Since the energy shift Δ_α is always positive in the energy region where the density of states increases with the energy, this produces a lowering of both z and β , indicated in Fig. 1 as solid circles (z^{dres} and β^{dres}).

Plugging the BED, Eq. (11), into Eq. (10) with the substitution $E^{\text{dres}} \rightarrow E$ returns the values of μ and β from which we can write down the corresponding BED indicated by solid curves in Fig. 2. Even if, in the case of weak interaction (top panels), the BED is hardly distinguishable from the “unperturbed one” for strong interaction (bottom panels), they are very different; nevertheless, they match the numerical data extremely well, without any fit.

The energy shifts Δ_α can be easily calculated numerically for each eigenstate. In Fig. 3 we plot such values, averaged over close eigenstates for the two different perturbation strengths V considered in Fig. 2.

It is also possible to derive an analytical expression for the energy shift Δ_α in Eq. (12) under not too strong assumptions.

For weak TBRI and a large number of particles, the form of the DOS is a Gaussian [15]. Moreover, due to the trace conservation of H , the position of the center E_c of the perturbed spectrum is the same as that of the unperturbed one. In this situation the variance σ_E of the perturbed DOS ρ is related to the variance σ_0 of the unperturbed DOS ρ_0 according to the simple relation (see Appendix B, Eq. (B15) for details)

$$\sigma_E^2 = \sigma_0^2 + \overline{(\Delta E)^2}, \quad (13)$$

where

$$\overline{(\Delta E)^2} = (1/N_H) \sum_n \sum_{k \neq n} H_{nk}^2 \quad (14)$$

is the average width of the SF and it can be obtained without any diagonalization. Inserting into Eq. (12) the spectral decomposition of H_0 ,

$$H_0 = \sum_k E_k^0 |k\rangle \langle k|, \quad (15)$$

one has

$$\Delta_\alpha = \sum_k E_k^0 |C_k^\alpha|^2 - E^\alpha \simeq \int dE (E - E^\alpha) \rho_0(E) \langle |C_k^\alpha|^2 \rangle. \quad (16)$$

Assuming a Gaussian form also for $\langle |C_k^\alpha|^2 \rangle$ peaking around E^α ,

$$\langle |C_k^\alpha|^2 \rangle \simeq \exp[-(E - E^\alpha)^2 / 2\overline{(\Delta E)^2}], \quad (17)$$

and for $\rho_0(E)$,

$$\rho_0(E) \simeq \exp[-(E - E_c)^2 / 2\sigma_0^2], \quad (18)$$

and inserting (17) and (18) into Eq. (16) with the correct normalizations, one gets the analytical estimate for the energy shift:

$$\Delta_\alpha = \frac{\overline{(\Delta E)^2}}{(\overline{(\Delta E)^2} + \sigma_0^2)} (E_c - E^\alpha). \quad (19)$$

These analytical values are shown in Fig. 3 as solid lines. As one can see, they work very well in the center of the energy spectrum [where the hypothesis of Gaussian local DOS (LDOS) and DOS can be applied without appreciable errors], while significant deviations appear at the low edge of the spectrum, where, due to the finite number of particles and levels, it is well known that DOS and LDOS cannot be described by Gaussians.

The increase in temperature ΔT , emerging due to the interparticle interaction, can be obtained from the definition of thermodynamic temperature by means of the unperturbed density of states ρ_0 ,

$$\beta = \frac{1}{T} = \frac{d \ln \rho_0}{dE},$$

so that

$$T^{\text{dres}} \equiv T + \Delta T = \left(\frac{d \ln \rho}{dE} \right)^{-1} = \frac{\sigma_E^2}{E_c - E} \quad (20)$$

and, finally, from Eq. (13)

$$\frac{\Delta T}{T} = \frac{\overline{(\Delta E)^2}}{\sigma_0^2}. \quad (21)$$

TABLE I. Parameters for Fig. 2. E^α is the eigenenergy, and β, z are the corresponding parameters of the BED with $E = E^\alpha$. E^{dres} is the corresponding dressed energy computed from the numerical value of Δ_α , and $\beta^{\text{dres}}, z^{\text{dres}}$ are the corresponding parameters of the BED with $E = E^{\text{dres}}$. $\Delta T/T$ is the relative temperature shift obtained from the dressed values, and $(\overline{\Delta E})^2/\sigma_0^2$ is the analytical value obtained from Gaussian approximation.

	E^α	$\beta = 1/T$	T	z	$E^{\text{dres}} = E^\alpha + \Delta_\alpha$	$\beta^{\text{dres}} = 1/T^{\text{dres}}$	T^{dres}	z^{dres}	$\Delta T/T$	$(\overline{\Delta E})^2/\sigma_0^2$
$V = 0.1$										
Fig. 2(a)	28.51	0.055	18.18	0.471	28.87	0.0525	19.05	0.464	0.048	0.031
Fig. 2(b)	11.27	0.2365	4.23	0.724	12.38	0.215	4.65	0.704	0.099	
$V = 0.4$										
Fig. 2(c)	25.93	0.0735	13.60	0.505	30.504	0.0417	23.98	0.446	0.763	0.51
Fig. 2(d)	6.05	0.434	2.30	0.873	16.59	0.156	6.41	0.637	1.787	

As one can see, the relative shift in temperature turns out to be independent of the eigenenergy E^α and dependent only on the ratio between the variance σ_0^2 of the unperturbed DOS and the average variance $(\overline{\Delta E})^2$ of the SF. Again, both can be found without the diagonalization of H . The analytical values of the temperature shift for the eigenstates in Fig. 2 agree fairly well with those obtained with the use of the energy shifts from Eq. (12) when the eigenenergy is far from the edges of the spectrum (top panels in Fig. 2). All numerical values for the shifts are reported for the reader's convenience in Table I.

Let us remark that even when the assumption of a Gaussian form of the DOS is not correct (for example, close to the

edges of the energy spectrum), the BED obtained with the dressed energy in Eq. (12) works very well, as clearly shown by a comparison between the solid curves and symbols in the bottom panels of Fig. 2.

IV. FLUCTUATIONS IN THE BOSE-EINSTEIN DISTRIBUTION

Above we have shown that, on average, the numerical data for n_s are in good agreement with the BED using a suitable dressed energy. However, in order to claim that statistical mechanics works on the level of individual states

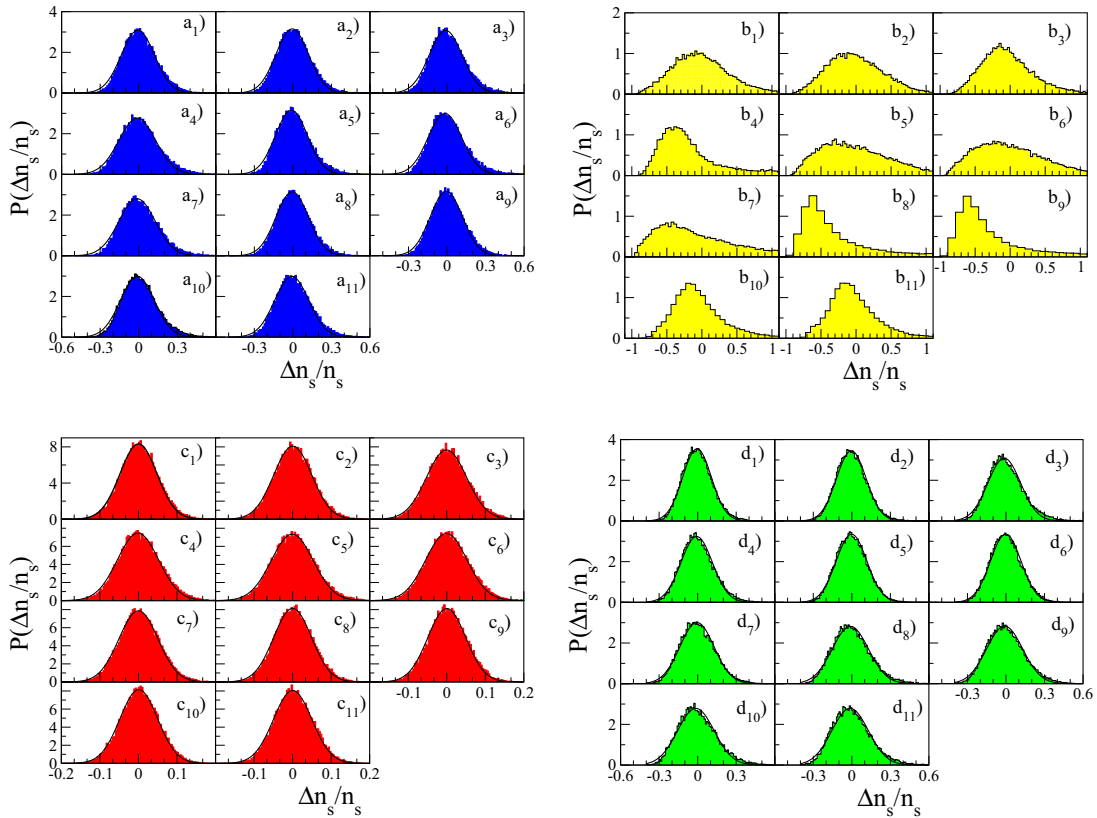


FIG. 4. Distributions of relative fluctuations $\Delta n_s/n_s, s = 1, \dots, M = 11$: (a) weak perturbation $V = 0.1$, high energy, (b) weak perturbation $V = 0.1$, low energy, (c) strong perturbation $V = 0.4$, high energy, and (d) strong perturbation $V = 0.4$, low energy. Statistics have been obtained with 10^3 different realizations of the random potential and by choosing different eigenstates in a small energy window in order to have approximately 20 eigenstates for each realization of the random potential. In all of them, where it is significant (all but the yellow distributions) we superimposed a Gaussian fit (black curve).

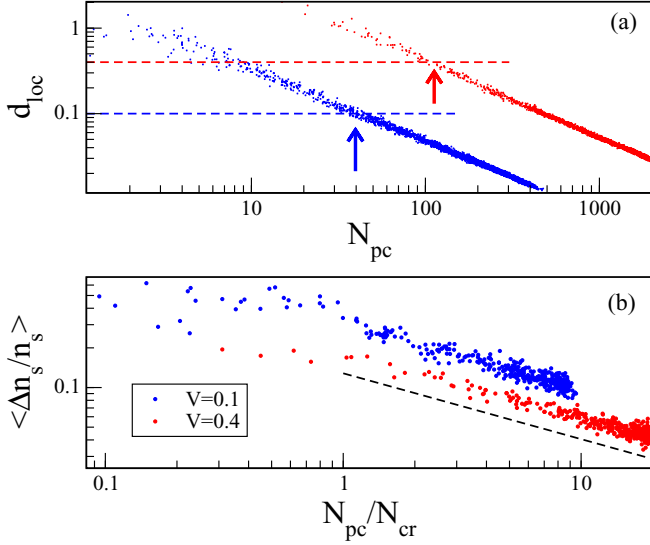


FIG. 5. (a) d_{loc} as a function of the number of principal components N_{pc} for each eigenstate $|\alpha\rangle$ and two different values of V : 0.1 (lower blue points) and 0.4 (upper red points). Dashed horizontal lines represent $d_{loc} = V$. Arrows define the critical values N_{cr} . (b) Average relative fluctuations in OND $\langle \Delta n_s / n_s \rangle$ vs N_{pc} / N_{cr} for two different V values as indicated in the legend. The dashed line is drawn to guide the eye and stands for $2/\sqrt{N_{pc}}$. Average fluctuations in OND have been obtained by averaging over 20 close eigenenergies.

one also has to check whether fluctuations are statistically acceptable. Fluctuations can emerge by varying the eigenstates in a small energy window or by different realizations of the disordered interparticle potential. We have checked that the distributions of these fluctuations are similar and can be considered statistically equivalent.

A study of fluctuations around average values is fundamental. Indeed, looking at the error bars in Fig. 2(b), it is clear that they can be very large, and one can wonder whether they can be considered statistically acceptable. Large fluctuations typically occur for eigenstates with energies close to the edges of the spectrum or for very weak interparticle interaction.

To face the question of how relevant the fluctuations are, we studied the distribution of the relative fluctuations

$$\frac{\Delta n_s}{n_s} \equiv \frac{n_s - \langle n_s \rangle}{\langle n_s \rangle}$$

of the occupation numbers for close (in energy) eigenstates. Results are shown in Fig. 4 for different s values, $s = 1, \dots, M$ and for the four eigenstates in Fig. 2.

These data clearly indicate that for all eigenstates except those in Fig. 2(b), whose distributions are in the top right panels of Fig. 4, we have that (i) fluctuations are independent of s and therefore statistically independent and (ii) they are approximately described by Gaussians, which is a strong result in view of the requirement of standard statistical mechanics.

Concerning the eigenstates used in Fig. 2(b), one can observe that for them one has $d_{loc} \approx 0.2 > V = 0.1$. Therefore, applying our local criterion for thermal chaotic eigenstates discussed above, we cannot treat them as chaotic eigenstates (in all other cases, of course, $V > d_{loc}$).

For a more quantitative analysis, for each eigenstate we have computed the corresponding value of d_{loc} as a function of the number of its principal components N_{pc} for the two values of V considered in Fig. 2. The intersections of these points with the horizontal lines $d_{loc} = V$, shown in Fig. 5(a), define the critical values N_{cr} indicated by arrows. Then, we expect the fluctuations in the OND to be not statistically acceptable when $N_{pc} < N_{cr}(V)$.

To test such a conjecture we group the eigenstates in small energy windows and calculate in each of them the average fluctuations in OND $\langle \Delta n_s / n_s \rangle$. In Fig. 5(b) we plot such a quantity vs the renormalized number of principal components N_{pc} / N_{cr} . As one can see, for $N_{pc} < N_{cr}$, the average fluctuations $\langle \Delta n_s / n_s \rangle$ are almost independent of N_{pc} , while for $N_{pc} > N_{cr}$ they decay as $1/\sqrt{N_{pc}}$ [dashed line in Fig. 5(b)]. This gives strong numerical evidence that for small systems and far from the thermodynamic limit, the value of N_{pc} plays the role of an “effective” number of particles.

V. CONCLUSIONS

We have shown that the standard Bose-Einstein distribution, known to appear for an ideal gas in the thermodynamic limit, emerges on the level of an individual eigenstate in an isolated system with a finite number of particles. This happens when the interparticle interaction is strong enough to lead to the onset of chaotic many-body eigenstates in the basis defined by the chosen single-particle spectrum. In our approach we gave an analytical threshold dependent on the eigenstates energy in order to have chaotic eigenstates. For those thermal eigenstates we computed the corresponding occupation number distribution and verified that they can be successfully described by a Bose-Einstein distribution with a suitable dressed energy dependent on the interparticle interaction. We also gave an analytical estimate for the dressed energy well confirmed by direct numerical data.

Special attention has been paid to the fluctuations of occupation numbers. We stress that in order to correspond to conventional statistical mechanics, fluctuations should be small, independent, and Gaussian. Specifically, they decrease as the square root of the number of principal components in chaotic eigenstates. Therefore, for finite isolated systems that are far from the thermodynamic limit, the control parameter for the onset of thermalization is not the number of particles but the number N_{pc} of principal components in chaotic eigenstates. Our analytical findings are complemented by numerical data.

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APPENDIX A: OCCUPATION NUMBER DISTRIBUTION FOR A SINGLE EIGENSTATE

It is important to observe that the occupation number distribution can be obtained for a single eigenstate, as claimed in the title. The only purpose of averaging over disorder or over close eigenstates as done in the main text was to introduce and

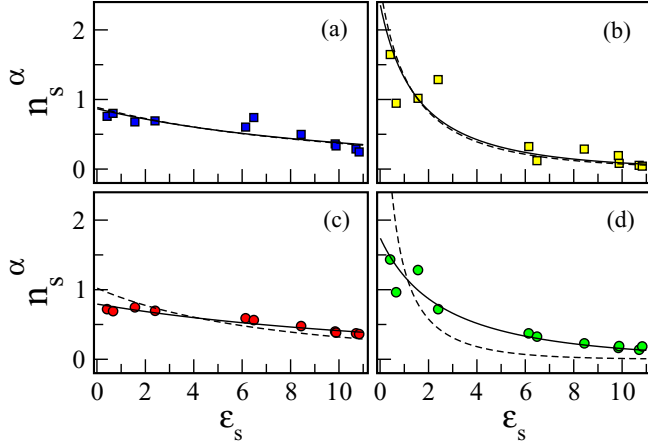


FIG. 6. Occupation numbers for a single eigenstate n_s^α for weak ($V = 0.1$, top panels) and strong ($V = 0.4$, bottom panels) perturbation and different energies in the middle of the energy spectrum, (a) $E^\alpha = 28.51$ and (c) $E^\alpha = 25.93$, and on the edges, (b) $E^\alpha = 11.27$ and (d) $E^\alpha = 6.05$. Dashed curves are obtained from the BED with $E = E^\alpha$ in Eq. (10). Solid curves correspond to the BED with the energy E^{dres} in Eq. (12).

analyze statistical errors. Examples of ONDs for four different eigenstates for the same parameters and energy regions as those in Fig. 2 are shown in Fig. 6.

APPENDIX B: PROPERTIES OF F FUNCTIONS AND THE DENSITY OF STATES

Let us start with the conventions used in the definitions of unperturbed (H_0) and perturbed ($H = H_0 + V$) Hamiltonians,

$$\begin{aligned} H_0|k\rangle &= E_k^0|k\rangle, \\ H|\alpha\rangle &= E^\alpha|\alpha\rangle, \end{aligned} \quad (\text{B1})$$

and the change of representation,

$$|\alpha\rangle = \sum_k C_k^\alpha |k\rangle. \quad (\text{B2})$$

The F function and the strength function, defined as

$$F^\alpha(E) = \sum_k |C_k^\alpha|^2 \delta(E - E_k^0) \quad (\text{B3})$$

and

$$F_k(E) = \sum_\alpha |C_k^\alpha|^2 \delta(E - E^\alpha), \quad (\text{B4})$$

satisfy many relations that are well known in the literature (see, for instance, Ref. [11]) and reported here for the reader's convenience. First of all, they are both normalized,

$$\int dE F^\alpha(E) = \int dE F_k(E) = 1. \quad (\text{B5})$$

Introducing the total density of states (DOS),

$$\rho(E) = \sum_\alpha \delta(E - E^\alpha), \quad (\text{B6})$$

and the unperturbed one,

$$\rho_0(E) = \sum_k \delta(E - E_k^0), \quad (\text{B7})$$

both normalized to the dimension of the Hilbert space N_H ,

$$\int dE \rho_0(E) = \int dE \rho(E) = N_H, \quad (\text{B8})$$

it is possible to write

$$F^\alpha(E) \simeq \rho_0(E) \langle |C_k^\alpha|^2 \rangle_{E_k^0=E}, \quad (\text{B9})$$

where the average is performed over a number of unperturbed eigenstates with energy close to E . In the same way, we have

$$F_k(E) \simeq \rho(E) \langle |C_k^\alpha|^2 \rangle_{E^\alpha=E}, \quad (\text{B10})$$

where the average is performed over a number of eigenstates with energy close to E . Note that instead of averaging over a number of eigenstates, one can use the average inside an individual eigenstate with the window moving method, provided this eigenstate has many uncorrelated components.

The two-body random interaction potential V is assumed to be noneffective on the diagonal, i.e.,

$$\langle k|V|k\rangle = 0.$$

From these simple relations we can get various results concerning the moments of the distributions.

1. First moment of the SF

The following equalities holds:

$$\begin{aligned} \langle E \rangle_k &= \sum_\alpha E^\alpha |C_k^\alpha|^2 = \sum_\alpha E^\alpha \langle k|\alpha\rangle \langle \alpha|k\rangle \\ &= \langle k|H|k\rangle = H_{kk} = \langle k|H_0|k\rangle = E_k^0. \end{aligned} \quad (\text{B11})$$

2. Second moment of the SF

$$\begin{aligned} (\Delta E_k)^2 &= \sum_\alpha (E^\alpha - \langle E \rangle_k)^2 |C_k^\alpha|^2 \\ &= \sum_\alpha (E^\alpha)^2 \langle k|\alpha\rangle \langle \alpha|k\rangle - \langle E \rangle_k^2 \\ &= \langle k|H^2|k\rangle - (H_{kk})^2 \\ &= \sum_j \langle k|H|j\rangle \langle j|H|k\rangle - (H_{kk})^2 \\ &= \sum_{j \neq k} H_{kj}^2. \end{aligned} \quad (\text{B12})$$

3. First moment (center) of the perturbed and unperturbed spectrum (DOS)

Let us define E_c as the center of the perturbed spectrum and E_c^0 as the center of the unperturbed one. It is easy to see

that in our model they coincide. Indeed,

$$\begin{aligned}\bar{E} = E_c &= \frac{1}{N_H} \sum_{\alpha} E^{\alpha} = \frac{1}{N_H} \text{Tr}[H] = \frac{1}{N_H} \sum_k \langle k|H|k\rangle \\ &= \frac{1}{N_H} \sum_k \langle k|H_0|k\rangle = \frac{1}{N_H} \sum_k E_k^0 = E_c^0.\end{aligned}\quad (\text{B13})$$

4. Second moment of the DOS

$$\begin{aligned}\overline{E^2} &= \frac{1}{N_H} \sum_{\alpha} E_{\alpha}^2 = \frac{1}{N_H} \text{Tr}[H^2] = \frac{1}{N_H} \sum_k \langle k|H^2|k\rangle \\ &= \frac{1}{N_H} \sum_k \sum_j \langle k|H|j\rangle \langle j|H|k\rangle \\ &= \frac{1}{N_H} \sum_k H_{kk}^2 + \frac{1}{N_H} \sum_k \sum_{j \neq k} H_{kj}^2.\end{aligned}\quad (\text{B14})$$

5. Relation between the second moments of the DOS

Let us define σ_E^2 and σ_0^2 as the variances of the perturbed and unperturbed DOS. Then, the following relation holds:

$$\begin{aligned}\sigma_E^2 &= \overline{E^2} - \bar{E}^2 = \frac{1}{N_H} \sum_k H_{kk}^2 + \frac{1}{N_H} \sum_k \sum_{j \neq k} H_{kj}^2 \\ &\quad - (E_c^0)^2 = \frac{1}{N_H} \sum_k (E_k^0)^2 - \left(\frac{1}{N_H} \sum_k E_k^0 \right)^2 \\ &\quad + \frac{1}{N_H} \sum_{j \neq k} (\Delta E_k)^2 \equiv \sigma_0^2 + \overline{(\Delta E)^2},\end{aligned}\quad (\text{B15})$$

where in the latter equality we have defined $\overline{(\Delta E)^2}$ as the average of the variances of all SF (each SF is defined for any given basis state $|k\rangle$).

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