Letter

Scale-invariant phase transition of disordered bosons in one dimension

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The disorder-induced quantum phase transition between superfluid and nonsuperfluid states of bosonic particles in one dimension is generally expected to be of the Berezinskii-Kosterlitz-Thouless (BKT) type. Here, we show that hard-core lattice bosons with power-law hopping decaying with distance as $1/r^\alpha$ with a finite integral over space—corresponding in spin language to an XY model with power-law couplings—undergo a non-BKT continuous phase transition instead. We use exact quantum Monte Carlo methods to determine the phase diagram for different values of the exponent α, focusing on the regime $\alpha > 2$. We find that the scaling of the superfluid stiffness with the system size is scale invariant at the transition point for any $\alpha \leqslant 3$—a behavior incompatible with the BKT scenario and typical of continuous phase transitions in higher dimension. By a scaling analysis near the transition point, we find that our data are consistent with a correlation length exponent satisfying the Harris bound $\nu \geqslant 2$ and demonstrate a universal behavior of disordered bosons in one dimension. For $\alpha > 3$ our data are consistent with a BKT scenario where the liquid is pinned by infinitesimal disorder.

Bosonic particles with local interactions in one dimension (1D) are described by a universal harmonic theory, known as a Luttinger liquid (LL). The latter corresponds to quantized superfluid hydrodynamics (including instantons) and is fully characterized by the superfluid velocity, $v = \sqrt{Y_s/\kappa}$, and LL parameter, $K = \pi\sqrt{\kappa Y_s}$, with κ the compressibility and Y_s the superfluid stiffness. Diagonal disorder induces an instability in LL towards a nonsuperfluid Bose glass (BG) phase—a compressible insulator displaying exponential decay of off-diagonal correlations. In their seminal paper [1], Giamarchi and Schulz found by means of a perturbative renormalization group (RG) analysis that the LL-BG transition is of the Berezinskii-Kosterlitz-Thouless (BKT) type that takes place at the universal value $K = K_c = 3/2$ (this result holds at the two-loop level [2]). In the strong-disorder limit, real-space RG treatments [3,4] and the "scratched-XY" criticality [5] also predict a BKT-type transition but at a nonuniversal value of $K_c > 3/2$. These considerations exhaust known scenarios for the disorder-induced superfluid to non-superfluid phase transitions in 1D.

In this Letter, we consider the disorder-induced localization transition in 1D superfluids of bosons with power-law hopping decaying with distance as $1/r^\alpha$. We utilize numerically exact large-scale quantum Monte Carlo simulations based on the worm algorithm [6] to determine the ground-state superfluid phases and phase transitions for different values of $\alpha > 2$. We find that the superfluid phases can be approximately characterized by an effective LL parameter K that reproduces the decay of correlation functions. However, contrary to existing theories, we find that the disorder-induced quantum phase transition is generically scale invariant and incompatible with the BKT scenario with the effective $K_c \leqslant 3/2$ for all $\alpha \leqslant 3$. As far as critical exponents are concerned, the data are consistent with the correlation length exponent

satisfying the Harris bound $\nu \geqslant 2$ for all values of $\alpha \leqslant 3$. Thus, our results reveal a universal behavior of bosons with power-law hopping in one dimension with a finite integral over space. For $\alpha > 3$ our results are instead consistent with a scenario where the superfluid is pinned by an infinitesimal disorder in the thermodynamic limit, similar to a BKT-like scenario for hard-core particles with short-range coupling. Our predictions are directly relevant for experiments with dipolar atoms and molecules, exciton materials, and cold ions.

We consider the following 1D lattice Hamiltonian for hard-core bosons

$$\mathcal{H} = -t \sum_{i<j} \frac{a^\alpha}{|r_{ij}|^\alpha}[b_i^\dagger b_j + \text{H.c.}] + \sum_i \epsilon_i n_i \quad (n_i \leqslant 1). \quad (1)$$

We employ standard notations for bosonic creation and annihilation operators on site i and restrict the maximal occupation number, $n_i = b_i^\dagger b_i$, to unity. The nearest-neighbor hopping amplitude t and the lattice spacing a are taken as units of energy and length, respectively. We choose random on-site energies ϵ_i uniformly distributed between $-W$ and W, and check that a different (Gaussian) choice of distribution does not affect the results. In spin language, Eq. (1) is equivalent to an XY Hamiltonian with power-law exchange couplings, which, in the absence of disorder, can be realized in experiments with cold polar molecules [7], trapped ions [8–10], and Rydberg atoms [11–16] (the latter can also be disordered [17]).

For an ideal system with $W/t = 0$, the spectra and low-energy phases of Hamiltonian (1) have been investigated by a variety of approaches. Using linear spin-wave theory, Ref. [18] identified $\alpha > 3$ as a regime where main properties reproduce those observed in the $\alpha = \infty$ limit of finite-range interactions, $1 < \alpha < 3$ as an intermediate regime with the
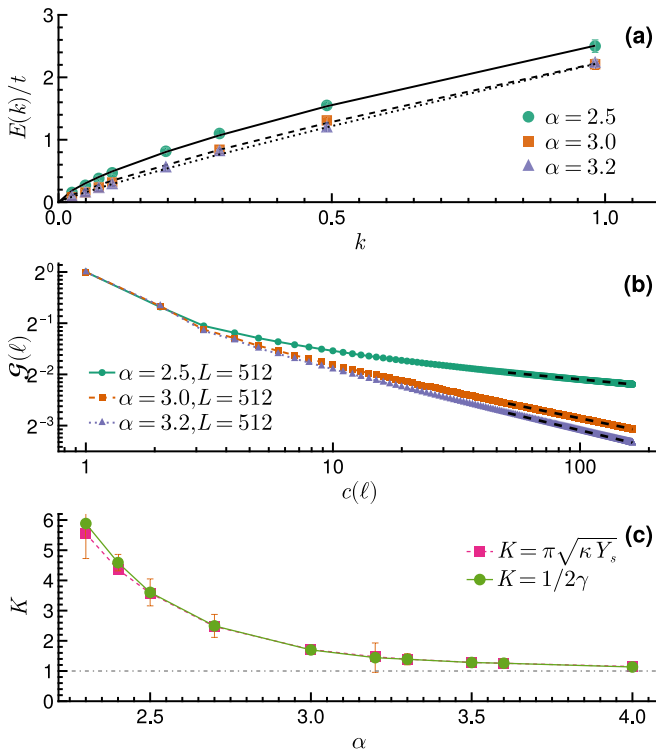
FIG. 1. Characterization of the superfluid phase for $W/t = 0$: (a) Dispersion relations $E(k)$ vs k for $\alpha = 2.5$, 3.0, and 3.2 chosen in the intermediate- and short-range regimes, respectively (see text) for $L = 256$. (b) Single-particle density matrix $\mathcal{G}(\ell)$ vs chord distance $c(\ell) = \sin(\pi\ell/L)/\sin(\pi/L)$ showing an algebraic decay for all α for $L = 512$. Dashed lines indicate the best fit with $Ac(\ell)^{-\gamma}$ where A, γ are fitting parameters. (c) Numerical evaluation of the Luttinger liquid parameter K as a function of α from the power-law decay $\mathcal{G} \propto \ell^{-1/(2K)}$ (green dots) and from the relation $K = \pi\sqrt{\kappa Y_s}$ (red squares) extrapolated to the thermodynamic limit via a polynomial scaling in $1/L$.

XY phase characterized by a continuously varying dynamical exponent $z = (\alpha - 1)/2$ (it governs the $k \to 0$ limit of the dispersion relation), and $\alpha < 1$ as a long-range regime with dispersionless excitations and properties similar to the infinite-range $\alpha = 0$ case in the thermodynamic limit. In this harmonic approach, $\alpha = 3$ is the boundary between the intermediate- and short-range regimes. Using a bosonization approach supplemented by an RG analysis, Ref. [19] predicts that power-law couplings are relevant in the RG sense for $\alpha < 3 - 1/(2K)$, with $K > 1$ to be determined numerically for each given α. In the following, we study the ground-state superfluid phases and phase transitions of Eq. (1) for $\alpha > 2$ using large-scale path-integral quantum Monte Carlo simulations based on the worm algorithm [6]. Without loss of generality, we focus on the particle density $\rho = 1/2$.

We start our analysis by first characterizing the bosonic liquid in the absence of disorder ($W/t = 0$). Figure 1(a) shows the dispersion relation $E(k)$ vs k for three values of $\alpha = 2.5$, 3.0, and 3.2 where k is the quasimomentum. It was deduced numerically from spectral peaks after analytic continuation of the imaginary-frequency dynamic structure factor [20,21]. The chosen values of α correspond to values

in the expected intermediate- ($\alpha = 2.5$), boundary- ($\alpha = 3.0$), and short-range ($\alpha = 3.2$) limits of the spin-wave analysis, respectively. The dispersion relation is nonlinear in k for $\alpha = 2.5$ (dots) and the data can be fit well by $E(k) \sim k^{z_*}$, with $z_* \simeq 0.74$, in good agreement with the $z = 0.75$ prediction of spin-wave analysis (continuous black line). In the short-range regime, instead, the dispersion relation is consistent with the linear law and a small negative quadratic contribution, also in agreement with literature. We checked that $E(k)$ for different system sizes agree with each other for the same values of k. (The dispersion relation in the superfluid phase for $\alpha = 2.5$ in the presence of disorder is sublinear, as presented in the Supplemental Material [22].)

For a 1D superfluid ground state, the single-particle density matrix $\mathcal{G}(\ell) = \langle b_i^\dagger b_{i+\ell} \rangle$ is expected to show an algebraic decay with the distance ℓ with diverging integral over space. Our data for $\mathcal{G}$ are shown in Fig. 1(b), for the same values of α as in Fig. 1(a) for a system with $L = 512$ sites and inverse temperature $\beta = L/t$. We observe algebraic decay $\mathcal{G} \sim \ell^{-\gamma}$ for all α.

Despite the nonlinear dispersion relation demonstrated above, we attempt a comparison with expectations from LL theory by extracting an effective LL parameter K as a function of α from two standard methods: the power-law decay $\mathcal{G} \sim \ell^{-\gamma}$ via the bosonization relation $\gamma = 1/(2K)$ (green dots) and the relation $K = \pi\sqrt{\kappa Y_s}$ (red squares). Both κ and Y_s can be conveniently computed by quantum Monte Carlo through the mean-square particle, N, and winding number, $\mathcal{W}$, fluctuations using the Pollock-Ceperley relation $Y_s = L\langle\mathcal{W}^2\rangle/\beta$. Figure 1(c) shows that the two methods produce similar estimates of K for all α, within the error bars, which is surprising, given the nonlinear dispersion relation demonstrated above. Moreover, K decreases monotonically and continuously with α from a large value $K \gtrsim 5$ at $\alpha \sim 2.3$ to $K \approx 1$ at $\alpha = 4$. Within an approximate LL scenario, this behavior may be explained by the fact that power-law hopping in Hamiltonian (1) allows for large-scale particle exchanges for small enough $\alpha < 3$, mimicking the behavior of soft-core bosons, for which one can easily get $K \gg 1$. The $K = 1$ value (dashed dotted line) corresponds to the short-range case of hard-core bosons with the nearest-neighbor hopping, a limit that is here asymptotically approached at $\alpha > 3$ [23]. In the following we analyze the situation at finite disorder strength and, in particular, explore the nature of the transition point, which is expected to be of the BKT type for Luttinger liquids. However, we note that this may not be the case here: BKT transition and its asymptotically exact RG flow are rooted in logarithmic interactions between vortex excitations. The latter originates from the kinetic energy of the flow around vortices $E \sim \int (n_s/m)dr/r$, where m is the particle mass and n_s the superfluid density. The single-particle spectrum in our model is not parabolic and formally corresponds to a scale-dependent "mass" $m(r) \sim r^{3-\alpha}$, implying that vortices in the superfluid phase should be bound by a power-law, not logarithmic, potential. It may thus be expected that BKT physics no longer applies for $\alpha \lesssim 3$.

We characterize the transition via the winding number fluctuations $\langle\mathcal{W}^2\rangle$ since they are a scale-invariant quantity, different from the superfluid stiffness Y_s. Figure 2(a) shows the evolution of superfluid properties measured by $\langle\mathcal{W}^2\rangle$ with
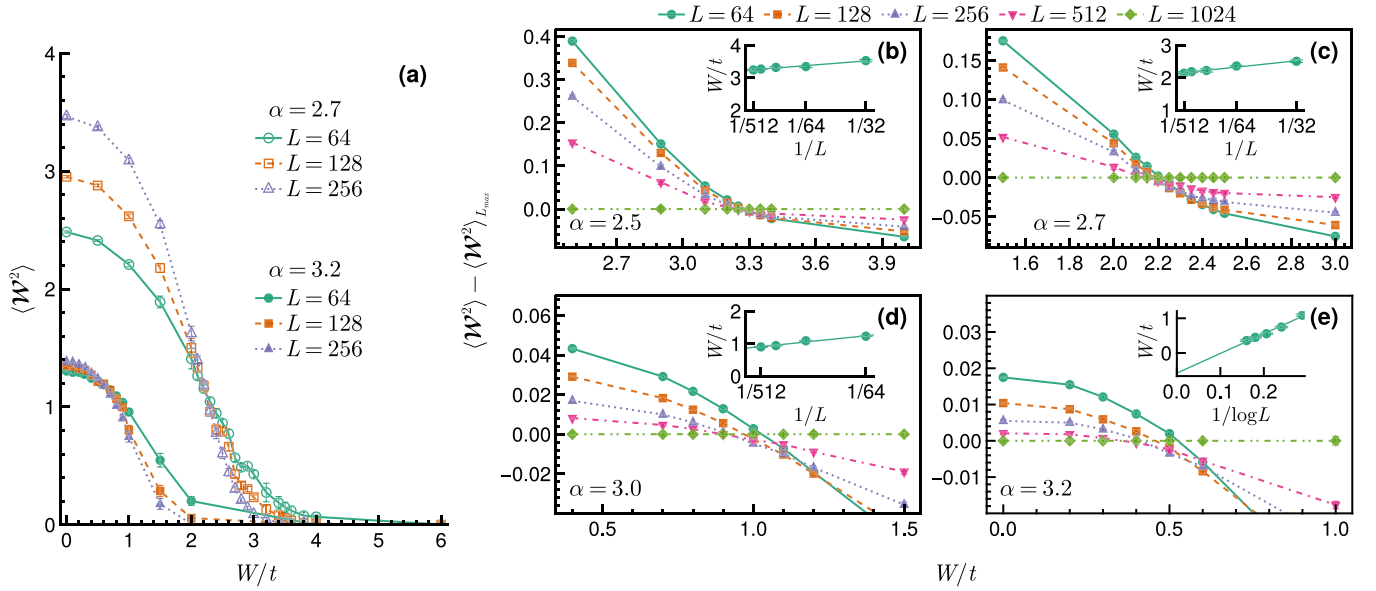
FIG. 2. Characterization of the superfluid to nonsuperfluid phase transition: (a) Mean-square winding number $\langle \mathcal{W}^2 \rangle$ vs disorder strength W/t for $\alpha = 2.7$ (open symbols) and 3.2 (solid symbols) for system sizes $L = 64, 128, 256$. (b)–(e) Zoom-in on the area near phase transitions for $\alpha = 2.5, 2.7, 3.0$, and 3.2, showing crossing between the curves; the curve corresponding to the largest size $L = 1024$ is subtracted from all data for clarity. Vertical error bars indicate the estimated uncertainty from the Monte Carlo simulations and disorder averages. Insets: Finite-size scaling of crossing points between curves for system sizes L_1 and $L_2 = 2L_1$ as a function of $L = L_1$.

disorder, W/t, for two example cases $\alpha = 2.7$ (open symbols) and 3.2 (solid symbols) and for several values of $L = 64, 128$, and 256. In both cases, $\langle \mathcal{W}^2 \rangle$ decrease monotonically with increasing W/t, until they reach near zero values. This behavior signals the transition between the superfluid and nonsuperfluid states. In the short-range case $\alpha = 3.2$, the behavior at larger values of disorder is reminiscent of what is expected for a BKT transition when in the infinite system $\langle \mathcal{W}^2 \rangle$ displays a jump to zero at the critical point [23]. However, surprisingly, for $\alpha = 2.7$ there is a clear crossing point of $\langle \mathcal{W}^2 \rangle$ around $W/t \sim 2$. This is inconsistent with the BKT criticality and is, instead, a signature of continuous scale-invariant phase transitions. This fact can be used to pinpoint the critical disorder strength W_c where superfluidity is lost by the crossing point of $\langle \mathcal{W}^2 \rangle$ vs W curves for different values of L. Figures 2(b)–2(e) present data in the vicinity of transition points for $\alpha = 2.5$, 2.7, 3.0, and 3.2 using $\beta = L/(8t)$ (even for $\alpha = 3.2$ our temperature is a factor of 2 smaller than the lowest phonon mode). Crossing points are very pronounced in Figs. 2(b) and 2(c) for intermediate exponents α, leaving no doubt that we are dealing with generic continuous transitions at $W/t = 3.24(5)$ for $\alpha = 2.5$ and at $W/t = 2.12(5)$ for $\alpha = 2.7$. The crossings appear to persist when transitioning to the short-range regime $\alpha \gtrsim 3$ [see Figs. 2(d) and 2(e) with crossings around $W/t = 0.87(5)$ for $\alpha = 3.0$ and around $W/t = 0.5(5)$ for $\alpha = 3.2$, contrary to all expectations]. However, a careful finite-size scaling up to large system sizes $L = 1024$ shows that the transition point for $\alpha > 3$ scales to $W/t \to 0$ in the thermodynamic limit [see the inset in Fig. 2(e), implying the absence of a continuous phase transition at finite W in the thermodynamic limit]. This is different from $\alpha \leqslant 3$, where the transition point scales to a finite value of W/t [see the insets in Figs. 2(b)–2(d)]. The breakdown of the BKT

scenario for all values $2 < \alpha \leqslant 3$ in Eq. (1) is surprising and is the main result of this Letter.

Figure 3(a) summarizes the ground-state phase diagram of Hamiltonian (1) in terms of W_c and α. Here, for each $\alpha \leqslant 3$, the critical point W_c is determined from the scale-invariant crossing point as described above. The critical disorder strength W_c/t decreases monotonically from a large
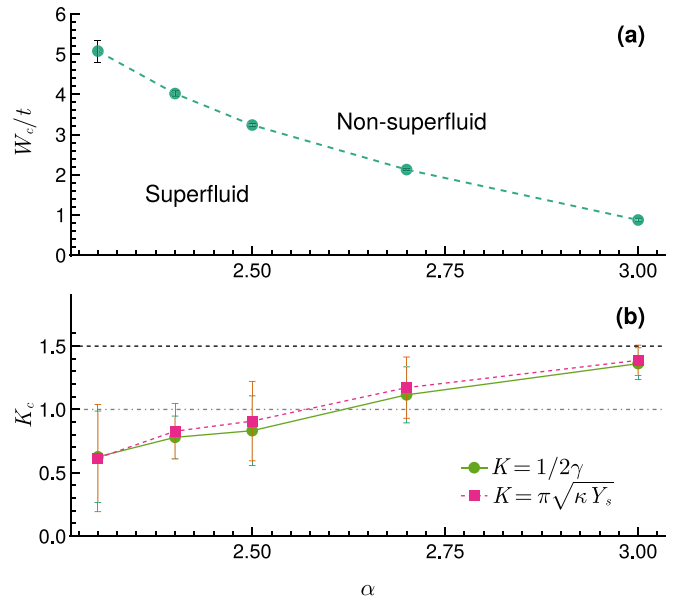

FIG. 3. (a) Phase diagram, W_c vs α, of the superfluid and nonsuperfluid quantum phases for model (1). (b) Critical values of LL parameter K_c at W_c vs α, as estimated from the power-law decay of $\mathcal{G}$ (green dots) and from $K = \pi \sqrt{\kappa Y_s}$ (red squares) for $L = 256$.
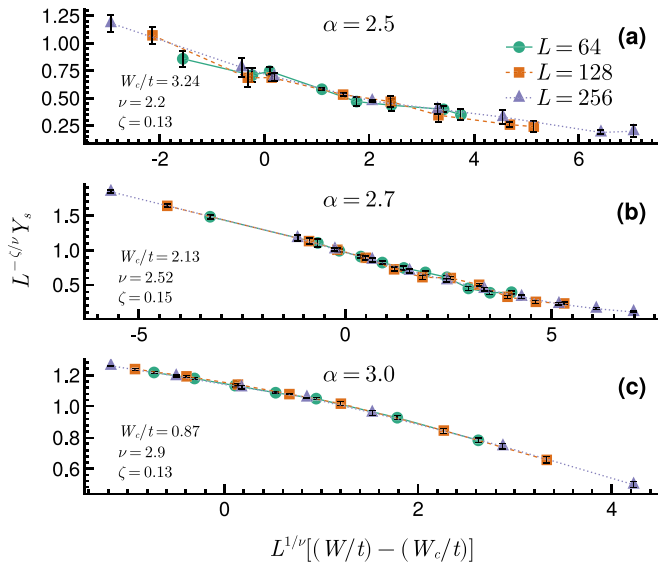
FIG. 4. (a)–(c) Data collapse for the scaled superfluid stiffness $L^{-\zeta/\nu}Y_s$ vs $L^{1/\nu}[(W/t)-(W_c/t)]$ for $\alpha=2.5$, 2.7, and 3.0 using $L=64$, 128, and 256. The fitted values of the correlation length exponent ν and ζ are reported directly in the figure. They satisfy $\nu\gtrsim 2$ for all α.
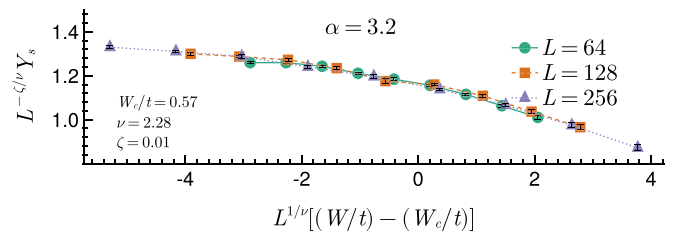

FIG. 5. Data collapse for the scaled superfluid stiffness $L^{-\zeta/\nu}Y_s$ vs $L^{1/\nu}[(W/t)-(W_c/t)]$ for $\alpha=3.2$ using $L=64$, 128, and 256. The fitted values of the correlation length exponent ν and ζ are reported directly in the figure. Unlike for $\alpha<3$, this scaling corresponds here to a finite-size effect, as the transition is not located at $W_c/t=0.57$ and slowly shifts to $W_c/t=0$ [see Fig. 2(e)].

value $W_c/t\sim 5.1$ to ~ 0.9 for $\alpha=3$. For $\alpha>3$ the transition in the thermodynamic limit occurs at $W_c/t=0^+$. The limiting value $W_c/t=0^+$ would correspond to the strictly short-range limit of hard-core bosons with short-range hopping, which are known to be localized by an infinitesimal disorder [23].

Figure 3(b) shows the critical LL parameter K_c computed at W_c/t for each value of α assuming that the approximate LL scenario properly describes the system. We find that, for $\alpha\lesssim 3$, K_c remains smaller than the critical BKT value of $3/2$ for short-range hopping models with weak disorder [23], confirming that the LL theory should not be used to describe the localization transition in the whole range $\alpha\leqslant 3$. In contrast, for $\alpha>3$ our results are in agreement with the conclusion that ideal systems with $K<3/2$ are ultimately pinned by disorder, leading to an insulating BG phase for any finite value of W/t. We complete our characterization of the quantum phase transition in Fig. 3(a) by determining the correlation length exponent ν using a data collapse analysis near the critical points [24]. For each α, the results of Monte Carlo simulations are rescaled by $L^{-\zeta/\nu}Y_s$ and collapsed on a single master curve using $L^{1/\nu}[(W/t)-(W_c/t)]$ as a variable. Critical values W_c/t are taken from the crossing points in Fig. 2, while ν and ζ are treated as fitting parameters and obtained using a Nelder-Mead algorithm [25] with a cost function based on the Kawashima-Ito-Houdayer-Hartmann quality metric [26,27]. Example results for $\alpha=2.5$, 2.7, 3.0 and $\alpha=3.2$ are shown in Figs. 4 and 5, respectively. We observe good collapse of all data near the critical points for $\alpha\geqslant 2$, and the obtained correlation length exponents always satisfy the so-called Harris bound $\nu\gtrsim 2$ (see Ref. [28]). In fact, this

result is expected from general arguments for a large class of d-dimensional disordered systems where an appropriately defined correlation length diverges [29]. Data collapse for $\alpha>3$ using $W_c/t=0.57$ is a finite-size effect given that this value of α is close to the boundary between the intermediate- and short-range regimes and crossing points slowly shift to zero with increasing the system size. However, this effect will likely be observed in experiments dealing with finite systems.

In conclusion, we have demonstrated that the disorder-induced superfluid to nonsuperfluid quantum phase transition for models with power-law hopping is a scale-invariant transition if $2<\alpha\leqslant 3$, ruling out the expected BKT scenario for interacting one-dimensional bosons in this regime. Our work opens up multiple other research directions, including whether the finite-temperature BKT scenario is generally inconsistent with power-law hopping models also in two dimensions [30–32]. Another open question is the nature of the nonsuperfluid quantum phase for general values of α. In Ref. [33] it was conjectured that for $\alpha=3$ this phase is a nonsuperfluid Bose metal phase with finite zero-frequency optical conductivity and algebraic decay of correlations. It is an open question whether similar behavior can be found for other α values. Our predictions should be directly testable in experiments for XY models realized via internal excitations of cold dipolar atoms and molecules, cold ion chains, and Rydberg atoms.

[1] T. Giamarchi and H. J. Schulz, Anderson localization and interactions in one-dimensional metals, Phys. Rev. B **37**, 325 (1988).

[2] Z. Ristivojevic, A. Petković, P. Le Doussal, and T. Giamarchi, Superfluid/Bose-glass transition in one dimension, Phys. Rev. B **90**, 125144 (2014).

[3] E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael, Insulating phases and superfluid-insulator transition of disordered boson chains, Phys. Rev. Lett. **100**, 170402 (2008).

[4] E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael, Superfluid-insulator transition of disordered bosons in one dimension, Phys. Rev. B **81**, 174528 (2010).

[5] L. Pollet, N. V. Prokof'ev, and B. V. Svistunov, Asymptotically exact scenario of strong-disorder criticality in one-dimensional superfluids, Phys. Rev. B **89**, 054204 (2014).

[6] N. V. Prokof'ev, B. Svistunov, and I. Tupitsyn, Exact, complete, and universal continuous-time worldline Monte Carlo approach to the statistics of discrete quantum systems, J. Exp. Theor. Phys. **87**, 310 (1998).

[7] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Observation of dipolar spin-exchange interactions with lattice-confined polar molecules, Nature (London) **501**, 521 (2013).

[8] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Non-local propagation of correlations in quantum systems with long-range interactions, Nature (London) **511**, 198 (2014).

[9] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, Quasiparticle engineering and entanglement propagation in a quantum many-body system, Nature (London) **511**, 202 (2014).

[10] L. Feng, O. Katz, C. Haack, M. Maghrebi, A. V. Gorshkov, Z. Gong, M. Cetina, and C. Monroe, Continuous symmetry breaking in a trapped-ion spin chain, Nature (London) **623**, 713 (2023).

[11] J. Zeiher, J.-Y. Choi, A. Rubio-Abadal, T. Pohl, R. van Bijnen, I. Bloch, and C. Gross, Coherent many-body spin dynamics in a long-range interacting Ising chain, Phys. Rev. X **7**, 041063 (2017).

[12] D. Barredo, H. Labuhn, S. Ravets, T. Lahaye, A. Browaeys, and C. S. Adams, Coherent excitation transfer in a spin chain of three Rydberg atoms, Phys. Rev. Lett. **114**, 113002 (2015).

[13] A. P. Orioli, A. Signoles, H. Wildhagen, G. Günter, J. Berges, S. Whitlock, and M. Weidemüller, Relaxation of an isolated dipolar-interacting Rydberg quantum spin system, Phys. Rev. Lett. **120**, 063601 (2018).

[14] A. Browaeys and T. Lahaye, Many-body physics with individually controlled Rydberg atoms, Nat. Phys. **16**, 132 (2020).

[15] G. Semeghini, H. Levine, A. Keesling, S. Ebadi, T. T. Wang, D. Bluvstein, R. Verresen, H. Pichler, M. Kalinowski, R. Samajdar, A. Omran, S. Sachdev, A. Vishwanath, M. Greiner, V. Vuletić, and M. D. Lukin, Probing topological spin liquids on a programmable quantum simulator, Science **374**, 1242 (2021).

[16] C. Chen, G. Bornet, M. Bintz, G. Emperauger, L. Leclerc, V. S. Liu, P. Scholl, D. Barredo, J. Hauschild, S. Chatterjee, M. Schuler, A. M. Läuchli, M. P. Zaletel, T. Lahaye, N. Y. Yao, and A. Browaeys, Continuous symmetry breaking in a two-dimensional Rydberg array, Nature (London) **616**, 691 (2023).

[17] S. de Léséleuc, V. Lienhard, P. Scholl, D. Barredo, S. Weber, N. Lang, H. P. Büchler, T. Lahaye, and A. Browaeys, Observation of a symmetry-protected topological phase of interacting bosons with Rydberg atoms, Science **365**, 775 (2019).

[18] I. Frérot, P. Naldesi, and T. Roscilde, Entanglement and fluctuations in the XXZ model with power-law interactions, Phys. Rev. B **95**, 245111 (2017).

[19] M. F. Maghrebi, Z.-X. Gong, and A. V. Gorshkov, Continuous symmetry breaking in 1D long-range interacting quantum systems, Phys. Rev. Lett. **119**, 023001 (2017).

[20] N. V. Prokof'Ev and B. V. Svistunov, Spectral analysis by the method of consistent constraints, JETP Lett. **97**, 649–653 (2013).

[21] O. Goulko, A. S. Mishchenko, L. Pollet, N. Prokof'ev, and B. Svistunov, Numerical analytic continuation: Answers to well-posed questions, Phys. Rev. B **95**, 014102 (2017).

[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.111.L020503 for the dispersion relation in the superfluid phase with finite disorder, which includes Refs. [20,21,34].

[23] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, UK, 2003).

[24] M. E. J. Newman and G. T. Barkema, *Monte Carlo Methods in Statistical Physics* (Oxford University Press, Oxford, UK, 1999).

[25] J. A. Nelder and R. Mead, A simplex method for function minimization, Comput. J. **7**, 308 (1965).

[26] N. Kawashima and N. Ito, Critical behavior of the three-dimensional $\pm J$ model in a magnetic field, J. Phys. Soc. Jpn. **62**, 435 (1993).

[27] J. Houdayer and A. K. Hartmann, Low-temperature behavior of two-dimensional Gaussian ising spin glasses, Phys. Rev. B **70**, 014418 (2004).

[28] A. B. Harris, Effect of random defects on the critical behaviour of Ising models, J. Phys. C **7**, 1671 (1974).

[29] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Finite-size scaling and correlation lengths for disordered systems, Phys. Rev. Lett. **57**, 2999 (1986).

[30] B. Sbierski, M. Bintz, S. Chatterjee, M. Schuler, N. Y. Yao, and L. Pollet, Magnetism in the two-dimensional dipolar XY model, Phys. Rev. B **109**, 144411 (2024).

[31] O. K. Diessel, S. Diehl, N. Defenu, A. Rosch, and A. Chiocchetta, Generalized Higgs mechanism in long-range-interacting quantum systems, Phys. Rev. Res. **5**, 033038 (2023).

[32] N. Defenu, T. Donner, T. Macrì, G. Pagano, S. Ruffo, and A. Trombettoni, Long-range interacting quantum systems, Rev. Mod. Phys. **95**, 035002 (2023).

[33] G. Masella, N. V. Prokof'ev, and G. Pupillo, Anti-Drude metal of bosons, Nat. Commun. **13**, 2113 (2022).

[34] N. Dupuis, Superfluid–Bose-glass transition in a system of disordered bosons with long-range hopping in one dimension, Phys. Rev. A **110**, 033315 (2024).

Supplemental Material for "Scale-invariant phase transition of disordered bosons in one dimension"

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In this work, we study a one-dimensional disordered lattice boson model with hopping amplitude decaying with distance as $1/r^\alpha$ using large scale quantum Monte-Carlo simulations. A recent work [1] based on approximate functional renormalization group methods (submitted to the archive after our work was published online) proposes that in the stable superfluid phase at finite disorder strength ($W/t > 0$), a density mode with linear dispersion (dynamical exponent $z = 1$) should emerge and the superfluid–Bose-glass transition follows the BKT universality class for any α, in contradiction to our results. In this Supplemental Material we check the predictions of [1]. Our analysis based on large scale numerical simulations confirms the picture that we present in the main text.

I. DISPERSION RELATION IN THE SUPERFLUID PHASE WITH FINITE DISORDER

The study presented in [1] proposes, using approximate functional renormalization group methods, that in the stable superfluid phase a density mode with linear dispersion (dynamical exponent $z = 1$) should emerge, and that the superfluid–Bose-glass transition follows the BKT universality class. To test these predictions, we performed large scale quantum Monte Carlo simulations for $\alpha = 2.5$ and disorder strength $W/t = 2.0$ to determine the dispersion relation by numerically analyzing the spectral peaks after performing an analytic continuation of the imaginary-frequency dynamic structure factor [2, 3].
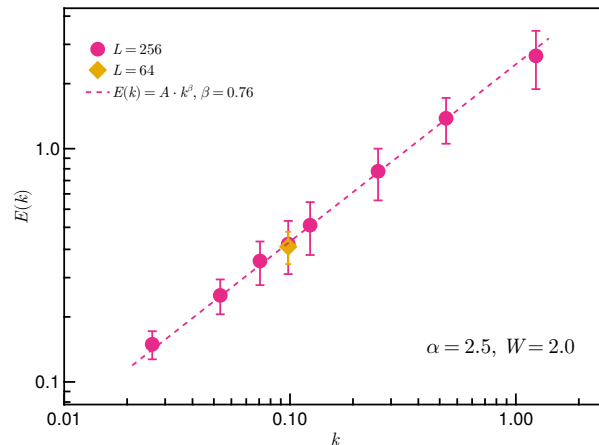

FIG. 1. Dispersion relation $E(k)$ vs k for $\alpha = 2.5$ in the superfluid phase at finite disorder strength $W/t = 2.0$.

Our results, presented in Figure 1 of this Supplemental Material, demonstrate that the spectrum is non-linear in k for $\alpha = 2.5$ at $W = 2.0$ and the data is well described by $E(k) \sim k^{z_*}$, with $z_* \simeq 0.76$. We checked for finite size effects (e.g., $L = 64$ in the figure), founding no significant impact. These results contradict the theoretical predictions of linear dispersion proposed in [1].

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[1] N. Dupuis, Phys. Rev. A **110**, 033315 (2024).
[2] N. V. Prokof'Ev and B. V. Svistunov, JETP Letters **97**, 649–653 (2013).
[3] O. Goulko, A. S. Mishchenko, L. Pollet, N. Prokof'ev, and B. Svistunov, Phys. Rev. B **95**, 014102 (2017).