Scale-invariant phase transition of disordered bosons in one dimension

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The disorder-induced quantum phase transition between superfluid and non-superfluid states of bosonic particles in one dimension is generally expected to be of the Berezinskii-Kosterlitz-Thouless (BKT) type. Here, we show that hard-core lattice bosons with integrable power-law hopping decaying with distance as $1/r^\alpha$ – corresponding in spin language to a XY model with power-law couplings – undergo a non-BKT continuous phase transition instead. We use exact quantum Monte-Carlo methods to determine the phase diagram for different values of the exponent α, focusing on the regime $\alpha > 2$. We find that the scaling of the superfluid stiffness with the system size is scale-invariant at the transition point for any $\alpha \leq 3$ – a behavior incompatible with the BKT scenario and typical of continuous phase transitions in higher dimension. By scaling analysis near the transition point, we find that our data are consistent with a correlation length exponent satisfying the Harris bound $\nu \geq 2$ and demonstrate a new universal behavior of disordered bosons in one dimension. For $\alpha > 3$ our data are consistent with a BKT scenario where the liquid is pinned by infinitesimal disorder.

Bosonic particles with local interactions in one dimension (1D) are described by a universal harmonic theory, known as Luttinger liquid (LL). The latter corresponds to quantized superfluid hydrodynamics (including instantons) and is fully characterized by the superfluid velocity, $v = \sqrt{Y_s/\kappa}$, and LL parameter, $K = \pi\sqrt{\kappa Y_s}$, with κ the compressibility and Y_s the superfluid stiffness. Diagonal disorder induces an instability in LL towards a non-superfluid Bose glass (BG) phase – a compressible insulator displaying exponential decay of off-diagonal correlations. In their seminal paper [1], Giamarchi and Schulz found by means of a perturbative renormalization group (RG) analysis that the LL-BG transition is of the Berezinskii-Kosterlitz-Thouless (BKT) type that takes place at the universal value $K = K_c = 3/2$ (this result holds at the two-loop level beyond the weak-disorder limit [2]). In the strong-disorder limit, real-space RG treatments [3, 4] and the "scratched-XY" criticality [5] also predict a BKT-type transition but at a non-universal value of $K_c > 3/2$. These considerations exhaust known scenarios for the disorder-induced superfluid to non-superfluid phase transitions in 1D.

In this work, we consider the disorder-induced localization transition in 1D superfluids of bosons with power-law hopping decaying with distance as $1/r^\alpha$. We utilize numerically exact large scale Quantum Monte-Carlo simulations based on the Worm Algorithm [6] to determine the ground-state superfluid phases and phase transitions for different values of $\alpha > 2$. We find that the superfluid phases can be accurately characterized by an effective LL parameter K that captures the decay of correlation functions. However, contrary to existing theories, we find that the disorder-induced quantum phase transition is generically scale-invariant and incompatible with the BKT scenario with the effective $K_c \leq 3/2$ for all $\alpha \leq 3$. As far as critical exponents are concerned, the data is consistent with the correlation length exponent satisfying the Harris bound $\nu \geq 2$ for all values of $\alpha \leq 3$. Thus, our results reveal a new universal behavior of bosons with integrable power-law hopping in one dimension. For $\alpha > 3$ our results are instead consistent with a scenario where the superfluid is pinned by an infinitesimal disorder in the thermodynamic limit, similar to a BKT-like scenario for hard-core particles with short-range coupling. Our predictions are directly relevant for experiments with dipolar atoms and molecules, exciton materials, and cold ions.

We consider the following 1D lattice Hamiltonian for hard-core bosons

$$\mathcal{H} = -t\sum_{i<j}\frac{a^\alpha}{|r_{ij}|^\alpha}\left[b_i^\dagger b_j + \text{H.c.}\right] + \sum_i \epsilon_i n_i, \qquad (n_i \leq 1). \tag{1}$$

We employ standard notations for bosonic creation and annihilation operators on site i and restrict the maximal occupation number, $n_i = b_i^\dagger b_i$, to unity. The nearest-neighbor hopping amplitude, t, and the lattice spacing, a, are taken as units of energy and length, respectively. Random on-site energies ϵ_i are uniformly distributed between $-W$ and W. In spin language, Eq. (1) is equivalent to an XY Hamiltonian with power-law exchange couplings, which, in the absence of disorder, can be realized in experiments with cold polar molecules [7], trapped ions [8–10] and Rydberg atoms [11–16] (the latter can also be disordered [17]).

For ideal system with $W/t = 0$, the spectra and low-energy phases of Hamiltonian (1) have been investigated by a variety of approaches. Using linear spin-wave theory, Ref. [18] identified $\alpha > 3$ as a regime where main properties reproduce those observed in the $\alpha = \infty$ limit of finite-range interactions; $1 < \alpha < 3$ as an intermediate regime with the XY phase characterized by a continuously varying dynamical exponent $z = (\alpha - 1)/2$ (it governs the $k \to 0$ limit of the dispersion relation); and $\alpha < 1$ as a long-range regime with dispersionless excitations and properties similar to the infinite-range $\alpha = 0$ case in the thermodynamic limit. In this harmonic approach, $\alpha = 3$ is the boundary between the intermediate and short-range regimes. Using a bosonization approach supplemented by an RG analysis, Ref. [19] predicts that power-law couplings are relevant in the RG sense for $\alpha < 3 - 1/(2K)$, with $K > 1$ to be determined numerically for each given α. In the following, we study the ground-state superfluid phases and phase transitions of Eq. (1) for $\alpha > 2$ using large scale path-integral quantum Monte-Carlo simulations based on the Worm algorithm [6]. Without loss of generality, we focus on the particle density $\rho = 1/2$.

We start our analysis by first characterizing the bosonic liquid in the absence of disorder ($W/t = 0$). Figure 1(a) shows the dispersion relation $E(k)$ vs k for three values of $\alpha = 2.5$, 3.0, and 3.2 where k is the quasi-momentum. It was deduced numerically from spectral peaks after analytic continuation of the imaginary-frequency dynamic structure factor [20, 21]. The chosen values of α correspond to values in the expected intermediate ($\alpha = 2.5$), boundary ($\alpha = 3.0$) and short-range ($\alpha = 3.2$) limits of the spin-wave analysis, respectively. The dispersion relation is non-linear in k for $\alpha = 2.5$ (dots) and the data can be fit well by $E(k) \sim k^{z_*}$, with $z_* \simeq 0.74$, in good agreement with the $z = 0.75$ prediction of spin-wave analysis (continuous black line). In the short-range regime, instead, the dispersion relation is consistent with the linear law and a small negative quadratic contribution. [Given restricted range of available k values, $\alpha = 3.0$ (squares) and 3.2 (triangles) results can be also fitted to the $E(k) \sim k^{z_*}$ law with exponents $z_* \simeq 0.82$ and $z_* \simeq 0.88$, respectively.]

For a 1D superfluid ground state, the single-particle density matrix $\mathcal{G}(\ell) = \langle b_i^\dagger b_{i+\ell} \rangle$ is expected to show a non-integrable algebraic decay with the distance ℓ. Our data for $\mathcal{G}$ are shown in Fig. 1(b), for the same values of α as in panel (a) for a system with $L = 256$ sites and inverse temperature $\beta = L$. We observe the algebraic decay $\mathcal{G} \sim \ell^{-\gamma}$ for all α. The exponent γ is then used below to extract numerically an effective LL parameter via the bosonization relation $\gamma = 1/(2K)$.

In order to directly compare with expectations from bosonization theory, in Fig. 1(c) we present the LL parameter as a function of α from two standard methods: the power-law decay $\mathcal{G} \sim \ell^{-\gamma}$ (green dots) and the relation $K = \pi\sqrt{\kappa Y_s}$ (red squares). Both κ and Y_s can be conveniently computed by quantum Monte Carlo through mean-square particle, N, and winding number, $\mathcal{W}$, fluctuations using the Pollock–Ceperley relation $Y_s = L\langle \mathcal{W}^2 \rangle/\beta$. Figure 1(c) shows that the two methods produce similar estimates of K for all α, within the error bars. Moreover, K decreases monotonically and continuously with α from a large value $K \gtrsim 5$ at $\alpha \sim 2.3$ to $K \approx 1$ at $\alpha = 4$. This behavior is explained by the fact that power-law hopping in Hamiltonian (1) allows for

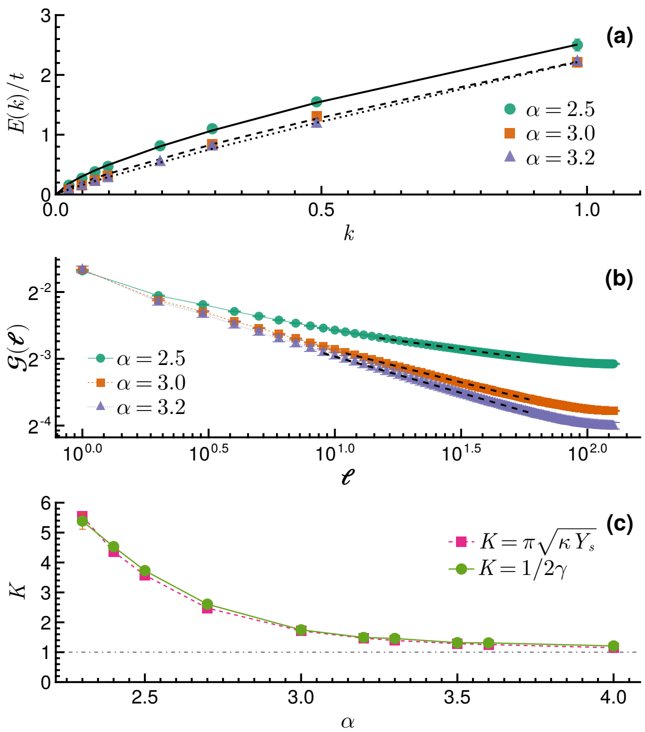

FIG. 1. Characterization of the superfluid phase for $W/t = 0$: (a) Dispersion relations $E(k)$ vs k for $\alpha = 2.5$, 3.0 and 3.2 chosen in the intermediate and short-range regimes, respectively (see text). (b) Single particle density matrix $\mathcal{G}(\ell)$ vs distance ℓ showing an algebraic decay for all α. (c) Numerical evaluation of the Luttinger liquid parameter K as a function of α from the power-law decay $\mathcal{G} \propto \ell^{-1/(2K)}$ (green dots) and from the relation $K = \pi\sqrt{\kappa Y_s}$ (red squares).

large-scale particle exchanges for small enough $\alpha < 3$, mimicking the behavior of soft-core bosons, for which one can easily get $K \gg 1$. The $K = 1$ value (dashed dotted line) corresponds to the short-range case of hard-core bosons with the nearest neighbor hopping, a limit that is here asymptotically approached at $\alpha > 3$. These results are overall consistent with expectations based on the spin-wave and bosonization theories [22, 23]: despite the unusual dispersion relations shown in panel (a) of Fig. 1, the liquid largely behaves as a regular LL with an effective Luttinger parameter K. In the following we analyse the situation at finite disorder strength and, in particular, explore the nature of the transition point.

Figure 2(a) shows the evolution of superfluid properties (measured through the winding number fluctuations $\langle \mathcal{W}^2 \rangle$) with disorder, W/t, for two example cases $\alpha = 2.7$ (empty symbols) and 3.2 (full symbols) and for several values of $L = 64, 128, 256$. In both cases, $\langle \mathcal{W}^2 \rangle$ decrease monotonically with increasing W/t, until they reach near zero values. This behavior signals the transition between the superfluid and non-superfluid states. In the short-range case $\alpha = 3.2$, the behavior at larger values of disorder is reminiscent of what is expected for a BKT

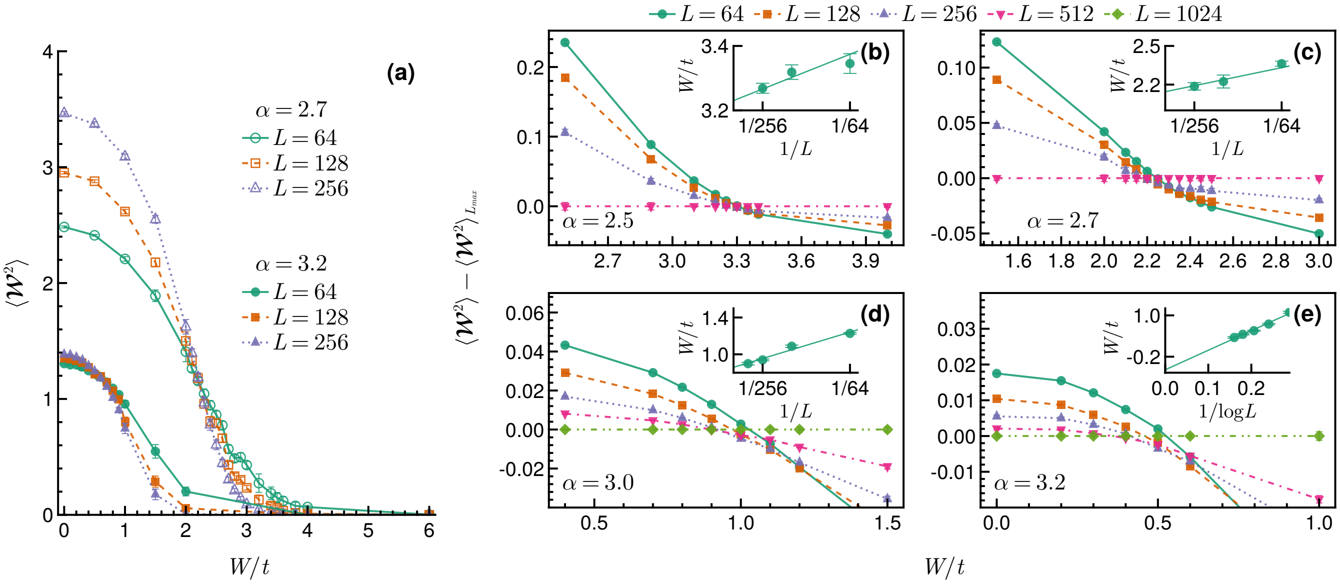

FIG. 2. Characterization of the superfluid to non-superfluid phase transition: (a) Mean-square winding number $\langle \mathcal{W}^2 \rangle$ vs disorder strength W/t for $\alpha = 2.7$ (empty symbols) and 3.2 (full symbols) for system sizes $L = 64$, 128, 256. (b)-(e) Zoom-in on the area near phase transitions for $\alpha = 2.5$, 2.7, 3.0 and 3.2, showing crossing between the curves; the curve corresponding to the largest size is subtracted from all data for clarity. Vertical error bars indicate the estimated uncertainty from the Monte Carlo simulations and disorder-averages. Insets: Finite-size scaling of crossings points between curves for system sizes L_1 and $L_2 = 2L_1$ as a function of $L = L_1$.

transition when in the infinite system $\langle \mathcal{W}^2 \rangle$ displays a jump to zero at the critical point [22]. However, surprisingly, for $\alpha = 2.7$ there is a clear crossing point of $\langle \mathcal{W}^2 \rangle$ around $W/t \sim 2$. This is inconsistent with the BKT criticality and is, instead, a signature of continuous scale-invariant phase transitions. This fact can be used to pinpoint the critical disorder strength W_c where superfluidity is lost by the crossing point of $\langle \mathcal{W}^2 \rangle$ -vs- W curves for different values of L. The panels (b)-(e) in Fig. 2 present data in the vicinity of transition points for $\alpha = 2.5, 2.7, 3.0$ and 3.2 using $\beta = L/8$ [even for $\alpha = 3.2$ our temperature is a factor of two smaller than the lowest phonon mode]. Crossing points are very pronounced in (b) and (c) for intermediate exponents α, leaving no doubt that we are dealing with generic continuous transitions at $W/t = 3.24(5)$ for $\alpha = 2.5$ and at $W/t = 2.12(5)$ for $\alpha = 2.7$. The crossings appear to persist when transitioning to the short-range regime $\alpha \gtrsim 3$, see panels (d) and (e) in Fig. 2 with crossings around $W/t = 0.87(5)$ for $\alpha = 3.0$ and around $W/t = 0.5(5)$ for $\alpha = 3.2$, contrary to all expectations. However a careful finite-size scaling up to large system sizes $L = 1024$ shows that the transition point for $\alpha > 3$ scales to $W/t \to 0$ in the thermodynamic limit, see Inset in Fig. 2(e), implying the absence of a continuous phase transition at finite W in the thermodynamic limit. The breakdown of the BKT scenario for all values $2 < \alpha \leq 3$ in Eq. (1) is surprising and is the main result of this work.

Figure 3(a) summarizes the ground state phase diagram of Hamiltonian (1) in terms of W_c and α. Here, for each $\alpha \leq 3$, the critical point W_c is determined from the scale-invariant crossing point as described above. The critical disorder strength W_c/t decreases monotonically

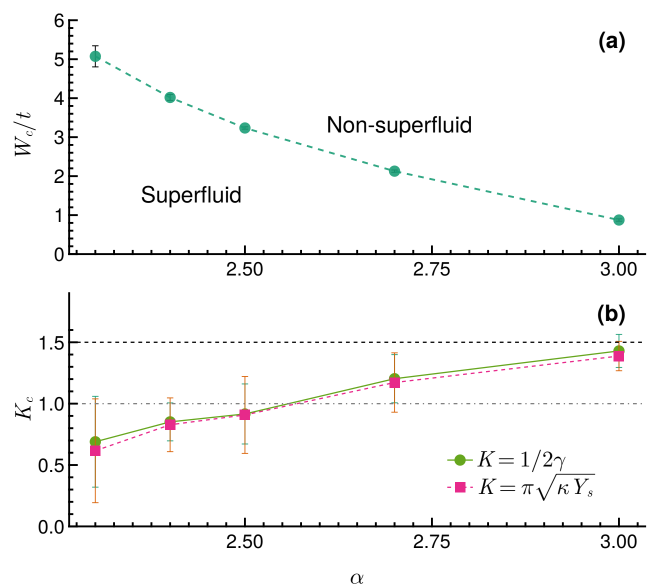

FIG. 3. (a) Phase diagram, W_c vs α, of the superfluid and non-superfluid quantum phases for model (1). (b) Critical values of LL parameter K_c vs α, as estimated from the power-law decay of $\mathcal{G}$ (green dots) and from $K = \pi\sqrt{\kappa Y_s}$ (red squares).

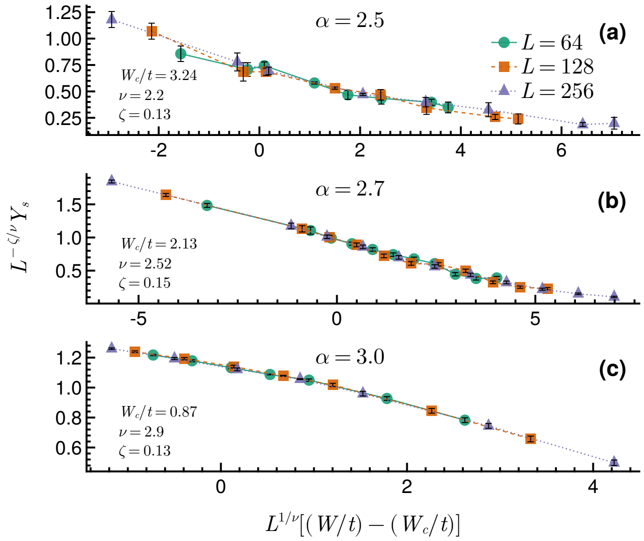

FIG. 4. (a)-(c) Data collapse for the scaled superfluid stiffness $L^{-\zeta/\nu}Y_s$ vs $L^{1/\nu}[(W/t)-(W_c/t)]$ for $\alpha = 2.5$, 2.7 and 3.0 using $L = 64$, 128, 256. The fitted values of the correlation length exponent ν and ζ are reported directly in the figure. They satisfy $\nu \gtrsim 2$ for all α.

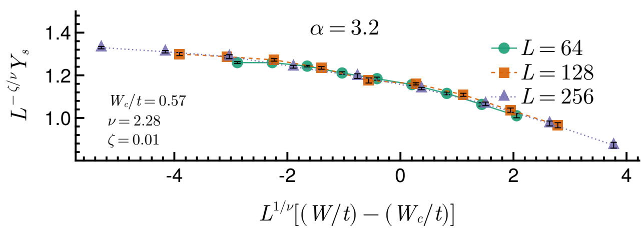

FIG. 5. Data collapse for the scaled superfluid stiffness $L^{-\zeta/\nu}Y_s$ vs $L^{1/\nu}[(W/t)-(W_c/t)]$ for $\alpha = 3.2$ using $L = 64$, 128, 256. The fitted values of the correlation length exponent ν and ζ are reported directly in the figure. Unlike for $\alpha < 3$, this scaling corresponds here to a finite-size effect, as the transition is not located at $W_c/t = 0.57$ and slowly shifts to $W_c/t = 0$, see Fig. 2(e).

from a large value $W_c/t \sim 5.1$ to ~ 0.9 for $\alpha = 3$. For $\alpha > 3$ the transition in the thermodynamic limit occurs at $W_c/t = 0^+$. The limiting value $W_c/t = 0^+$ would correspond to the strictly short-range limit of hard-core bosons with short-range hopping, which are known to be localized by an infinitesimal disorder [22].

Figure 3(b) shows the critical LL parameter K_c computed at W_c/t for each value of α. Contrary to previous theories, here we find that, for $\alpha \lesssim 3$, K_c remains smaller than the critical BKT value of 3/2 for short-range hopping models with weak disorder [22]. The behavior for $\alpha \lesssim 3$ is thus incompatible with known results from bozonisation theory.

In contrast, for $\alpha > 3$ our results are in agreement with conclusion that ideal systems with $K < 3/2$ are ultimately pinned by disorder, leading to an insulating BG phase for any finite value of W/t.

We complete our characterization of quantum phase transition in Fig. 3(a) by determining the correlation length exponent ν using data collapse analysis near the critical points [24]. For each α, the results of Monte Carlo simulations are rescaled by $L^{-\zeta/\nu}Y_s$ and collapsed on a single master curve using $L^{1/\nu}[(W/t)-(W_c/t)]$ as a variable. Critical values W_c/t are taken from the crossing points in Fig. 2, while ν and ζ are treated as fitting parameters and obtained using a Nelder-Mead algorithm [25] with a cost function based on the Kawashima-Ito-Houdayer-Hartmann quality metric [26, 27]. Example results for $\alpha = 2.5$, 2.7, 3.0 and $\alpha = 3.2$ are shown in Figs. 4 and 5, respectively. We observe good collapse of all data near the critical points for $\alpha \geq 2$, and the obtained correlation length exponents always satisfy the so-called Harris bound $\nu \gtrsim 2$, see [28]. In fact, this result is expected from general arguments for a large class of d-dimensional disordered systems where an appropriately defined correlation length diverges [29]. Data collapse for $\alpha > 3$ using $W_c/t = 0.57$ is a finite-size effect given that this value of α is close to the boundary between the intermediate and short-range regimes and crossing points slowly shift to zero with increasing the system size. However, this effect will likely be observed in experiments dealing with finite systems.

In conclusion, we have demonstrated that the disorder-induced superfluid to non-superfluid quantum phase transition for models with power-law hopping is a scale-invariant transition if $2 < \alpha \leq 3$, ruling out the expected BKT scenario for interacting one-dimensional bosons in this regime. Our work opens up multiple other research directions, including whether the finite-temperature BKT scenario is generally inconsistent with power-law hopping models also in two dimensions [23, 30, 31]. Another open question is the nature of the non-superfluid quantum phase for general values of α. In Ref. [32] it was conjectured that for $\alpha = 3$ this phase is a non-superfluid Bose metal phase with finite zero-frequency optical conductivity and algebraic decay of correlations. It is an open question whether similar behavior can be found for other α values. Our predictions should be directly testable in experiments for XY models realized via internal excitations of cold dipolar atoms and molecules, cold ions chains, and Rydberg atoms.

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